INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2009; **61**:698–708
Published online 4 December 2008 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.1973

SHORT COMMUNICATION

On series solution for generalized Falkner–Skan flow of a FENE-P model

S. Abbasbandy[1,2,*,†,‡], T. Hayat[2] and K. Maqbool[2]

[1]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran 14778, Iran*
[2]*Department of Mathematics, Quaid-i-Azam University, Islamabad 44000, Pakistan*

SUMMARY

This paper deals with the viscoelastic boundary layer flow past a plate. Constitutive assumptions of the FENE-P model are taken into account. The pressure gradient is taken as non-zero. The series solution of the non-linear problem modelled in (*Appl. Math. Lett.* 2007; **20**:1211–1215) is developed by a homotopy analysis method (HAM). Numerical solution of the skin friction coefficient is also computed. Further a comparison between the numerical and HAM solutions is provided. Copyright © 2008 John Wiley & Sons, Ltd.

Received 4 August 2008; Revised 22 October 2008; Accepted 26 October 2008

KEY WORDS: Falkner–Skan equation; boundary layer; FENE-P fluid; homotopy analysis method (HAM); Crocco's transformation; Adomian's method

1. INTRODUCTION

Investigation of boundary layer flows is the subject of great interest because of their engineering and industrial applications. In the past the boundary layer theory has been widely used to various flow equations of non-Newtonian fluid models. Among the many fluid models, an interesting one is the FENE-P model. Several investigations using FENE-P model are made by many investigators including Agarwala and O'Regan [1], Olagunju [2], Asaithambi [3] and several references therein. Very recently, Anabtawi and Khuri [4] found numerical solutions for generalized Falkner–Skan

*Correspondence to: S. Abbasbandy, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran 14778, Iran.
†E-mail: abbasbandy@yahoo.com
‡Permanent address: Department of Mathematics, Imam Khomeini International University, Ghazvin 34149-16818, Iran.

Contract/grant sponsor: Higher Education Commission (HEC) of Pakistan

flow of a FENE-P fluid. Alizadeh *et al*. [5] also found the solution of the Falkner–Skan equation for wedge by utilizing Adomian decomposition method.

The purpose of present attempt is two fold. First to find a series solution of the non-linear problem given in study [4] by using homotopy analysis method (HAM) [6–16]. The numerical solution for the skin friction is also tabulated. A comparison between the solutions obtained is presented.

2. PROBLEM STATEMENT

The generalized Falkner–Skan flows problem in a FENE-P fluid is [2, 4]

$$(Hf'')' + ff'' + \frac{2m}{m+1}(1-(f')^2) = 0 \tag{1}$$

$$f(0) = f_\omega, \quad f'(0) = 0, \quad f'(\infty) = 1 \tag{2}$$

where $f = f(\eta)$ satisfies

$$\Lambda g^3(\eta)(f'')^2 + g(\eta) - \gamma = 0$$

Here η is the dimensionless variable and f_ω is a constant that corresponds to blowing/injection or suction. In addition, $\gamma = b/(3+b)$, where b is the extensibility parameter, and

$$\Lambda = \frac{(m+1)\varepsilon}{3+b}$$

where $\varepsilon = We^2 Re$. The constant We is the well-known Weissenberg number, which is the measure of the fluid elasticity and Re is the Reynolds number, see [4] for more details.

As [4], in this study we consider the case where $\varepsilon \ll 1$. In addition, the variable H in Equation (1) is given by

$$H = 1 - \beta + \beta g$$

where β is the retardation parameter. By these assumptions, in [4], it is proved that $H \neq 0$ is constant and we have the following generalized Falkner–Skan equation:

$$Hf''' + ff'' + \frac{2m}{m+1}(1-(f')^2) = 0 \tag{3}$$

Now, by introducing the transformations

$$f(\eta) = \sqrt{H} F(\xi), \quad \xi = \frac{\eta}{\sqrt{H}} \tag{4}$$

we have the following problem:

$$F'''(\xi) + F(\xi)F''(\xi) + \alpha(1-(F'(\xi))^2) = 0 \tag{5}$$

$$F(0) = \frac{f_\omega}{\sqrt{H}}, \quad F'(0) = 0, \quad F'(\infty) = 1 \tag{6}$$

where $\alpha = 2m/(m+1)$.

DOI: 10.1002/fld

3. CROCCO'S TRANSFORMATION

A direct solution of Equation (5) with boundary conditions (6) can be obtained by a shooting method using Runge–Kutta algorithm or other iterative numerical methods, see, for example, [17–20]. The arising difficulty is that we have to make an initial guess for the value $F''(0)$ to initiate the shooting process and this guess is very important to obtain a good solution. Unfortunately, the process is very sensitive to this starting value and the problem is made worse by the values of α and $F(0)$ in the problem.

Crocco's transformation can be used in calculating $F''(0)$. The idea is to transform the equation and boundary conditions into another set of variables. Choose a suitable profile for the unknown function and then integrate over the complete range of the independent variable [21]. We first introduce the transformations

$$\lambda = F'(\xi), \quad \psi = (F''(\xi))^2 \tag{7}$$

which convert Equation (5) into the following second-order differential equation:

$$\psi \frac{\mathrm{d}^2\psi}{\mathrm{d}\lambda^2} - \frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\lambda}\right)^2 + \alpha(\lambda^2-1)\frac{\mathrm{d}\psi}{\mathrm{d}\lambda} - (4\alpha-2)\lambda\psi = 0 \tag{8}$$

The boundary conditions (6) together with the fact that $F''(+\infty)=0$ give the following conditions for Equation (8):

$$\lambda = 0, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\lambda} = -2(aM+\alpha) \tag{9}$$

$$\lambda = 1, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\lambda} = 0 \tag{10}$$

where $a = F(0) = f_\omega/\sqrt{H}$ and $M = F''(0)$. Further, from Equations (8)–(10) we obtain the so-called supplementary boundary conditions:

$$\lambda = 1; \quad \psi = 0 \tag{11}$$

$$\lambda = 0; \quad \psi = M^2 \tag{12}$$

To solve Equation (8) subject to the conditions (9)–(12), we choose the following profile for ψ:

$$\psi(\lambda) = (aM+\alpha)(-2\lambda^3+4\lambda^2-2\lambda) + M^2(2\lambda^3-3\lambda^2+1) \tag{13}$$

which satisfies (9)–(12). Substituting Equation (13) into Equation (8) and integrating with respect to λ from $\lambda=0$ to $\lambda=1$, we obtain a four-order polynomial for M. We take the algebraically larger root as the other root does not give realistic results. Hence, in this method, calculating M for various values of α and $F(0)$ is very easy and indeed we can find that the problem for which values of these parameters are realistic.

DOI: 10.1002/fld

4. SOLUTION BY HOMOTOPY ANALYSIS METHOD (HAM)

According to Equation (5) and the boundary conditions (6), solution can be expressed in the form

$$F(\xi)=\sum_{q,n=0}^{+\infty} c_{q,n}\xi^q \mathrm{e}^{-n\delta\xi} \tag{14}$$

where the $c_{q,n}(q,n=0,1,\ldots)$ are coefficients to be determined and $\delta>0$ is a spatial-scale parameter. According to the *rule of solution expression* denoted by Equation (14) and the boundary conditions (6), it is natural to choose

$$F_0(\xi)=a+\xi-\frac{1-\exp(-\delta\xi)}{\delta} \tag{15}$$

as the initial approximation to $F(\xi)$, where $a=F(0)=f_\omega/\sqrt{H}$. We define an auxiliary linear operator $\mathscr{L}$ by

$$\mathscr{L}[\phi(\xi,\delta;p)]=\left(\frac{\partial^3}{\partial\xi^3}+\delta\frac{\partial^2}{\partial\xi^2}\right)\phi(\xi,\delta;p) \tag{16}$$

with the property

$$\mathscr{L}[C_1+C_2\xi+C_3\mathrm{e}^{-\delta\xi}]=0 \tag{17}$$

where C_1, C_2 and C_3 are constants. This choice of $\mathscr{L}$ is motivated by Equation (14) and the later requirement that (25) should contain only two non-zero constants, namely C_1 and C_3.

From (5) we define a non-linear operator

$$\mathscr{N}[\phi(\xi,\delta;p)]=\left(\frac{\partial^3\phi}{\partial\xi^3}\right)+\phi\left(\frac{\partial^2\phi}{\partial\xi^2}\right)+\alpha\left[1-\left(\frac{\partial\phi}{\partial\xi}\right)^2\right] \tag{18}$$

and then construct the homotopy

$$\mathscr{H}[\phi(\xi,\delta;p)]=(1-p)\mathscr{L}[\phi(\xi,\delta;p)-F_0(\xi)]-\hbar p\mathscr{N}[\phi(\xi,\delta;p)] \tag{19}$$

where $\hbar\neq 0$ is the convergence-control parameter [16]. Setting $\mathscr{H}[\phi(\xi,\delta;p)]=0$, we have the zero-order deformation problem as follows:

$$(1-p)\mathscr{L}[\phi(\xi,\delta;p)-F_0(\xi)]=\hbar p\mathscr{N}[\phi(\xi,\delta;p)] \tag{20}$$

$$\phi(0,\delta;p)=a,\quad \left.\frac{\partial\phi(\xi,\delta;p)}{\partial\xi}\right|_{\xi=0}=0,\quad \left.\frac{\partial\phi(\xi,\delta;p)}{\partial\xi}\right|_{\xi=+\infty}=1 \tag{21}$$

where $p\in[0,1]$ is an embedding parameter. When the parameter p increases from 0 to 1, the solution $\phi(\xi,\delta;p)$ varies from $F_0(\xi)$ to $F(\xi)$. If this continuous variation is smooth enough, the Maclaurin's series with respect to p can be constructed for $\phi(\xi,\delta;p)$, and further, if this series is convergent at $p=1$, we have

$$F(\xi)=F_0(\xi)+\sum_{n=1}^{+\infty}F_n(\xi)=\sum_{n=0}^{+\infty}\varphi_n(\xi,\hbar,\delta) \tag{22}$$

DOI: 10.1002/fld

where

$$F_n(\xi)=\frac{1}{n!}\frac{\partial^n\phi(\xi,\delta;p)}{\partial p^n}\bigg|_{p=0}$$

Differentiating Equations (20) and (21) n times with respect to p, then setting $p=0$, and finally dividing by $n!$, we obtain the nth-order deformation problem

$$\mathscr{L}[F_n(\xi)-\chi_n F_{n-1}(\xi)]=\hbar R_n(\xi)\quad (n=1,2,3,\ldots) \tag{23}$$

$$F_n(0)=0,\quad F_n'(0)=0,\quad F_n'(+\infty)=0 \tag{24}$$

where R_n is defined as

$$R_n(\xi)=F'''_{n-1}+\sum_{i=0}^{n-1}F_i F''_{n-i-1}-\alpha\sum_{i=0}^{n-1}F_i' F'_{n-i-1}+\alpha(1-\chi_n)$$

with

$$\chi_n=\begin{cases}0, & n\leqslant 1\\ 1, & n>1\end{cases}$$

The general solution of Equation (23) is

$$F_n(\xi)=\hat{F}_n(\xi)+C_1+C_2\xi+C_3\mathrm{e}^{-\delta\xi} \tag{25}$$

where C_1, C_2 and C_3 are constants and $\hat{F}_n(\xi)$ is a particular solution of Equation (23).

Using $F_n'(+\infty)=0$, we have $C_2=0$. The unknowns C_1 and C_3 are governed by

$$\hat{F}_n(0)+C_1+C_3=0,\quad \hat{F}_n'(0)-\delta C_3=0$$

In this way, we derive $F_n(\xi)$ for $n=1,2,3,\ldots$, successively. At the Nth-order approximation, we have the analytic solution of Equation (5), namely

$$F(\xi)\approx\sum_{n=0}^{N}F_n(\xi) \tag{26}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (26) in the HAM. By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$, which ensures the convergence of the solution series. As pointed out by Liao [6], the appropriate region for $\hbar$ is a horizontal line segment.

5. NUMERICAL RESULTS

We use the widely applied symbolic computation software MATHEMATICA to solve Equations (23) and find that $\varphi_n(\xi,\hbar,\delta)$ has the following structure:

$$\varphi_n(\xi,\hbar,\delta)=\sum_{k=0}^{n+1}\Psi_{n,k}(\xi,\hbar,\delta)\exp(-k\delta\xi),\quad n\geqslant 0$$

DOI: 10.1002/fld

where the function $\Psi_{n,k}(\xi,\hbar,\delta)$ is defined by

$$\Psi_{0,0}(\xi,\hbar,\delta)=b_{0,0}^0+b_{0,0}^1\xi$$

$$\Psi_{0,1}(\xi,\hbar,\delta)=b_{0,1}^0$$

$$\Psi_{n,0}(\xi,\hbar,\delta)=b_{n,0}^0,\quad n\geqslant 1$$

$$\Psi_{n,k}(\xi,\hbar,\delta)=\sum_{i=0}^{2(n+1-k)} b_{n,k}^i\xi^i,\quad n\geqslant 1,\ \ 1\leqslant k\leqslant n+1$$

and the related coefficients are

$$b_{0,0}^0=a-\frac{1}{\delta},\quad b_{0,0}^1=1,\quad b_{0,1}^0=\frac{1}{\delta}$$

$$b_{1,0}^0=\hbar\left[\frac{-5}{4\delta^3}-\frac{7\alpha}{4\delta^3}-\frac{a}{\delta^2}+\frac{1}{\delta}\right]$$

$$b_{1,1}^0=\hbar\left[\frac{3}{2\delta^3}+\frac{3\alpha}{2\delta^3}+\frac{a}{\delta^2}-\frac{1}{\delta}\right]$$

$$b_{1,1}^1=\hbar\left[-1+\frac{a}{\delta}+\frac{1}{\delta^2}+\frac{2\alpha}{\delta^2}\right]$$

$$b_{1,1}^2=\frac{\hbar}{2\delta}$$

$$b_{1,2}^0=\hbar\left[\frac{-1}{4\delta^3}+\frac{\alpha}{4\delta^3}\right]$$

and so on. Note that the infinite series (22) gives a family of explicit analytic solutions in two parameters δ ($\delta>0$) and $\hbar$ ($\hbar\neq 0$). Note that the HAM provides us with great freedom and large flexibility to select better values of δ and $\hbar$ so as to ensure that the related series (22) converges to $F(\xi)$. Certainly, if Equation (22) converges, its second-order derivative with respect to ξ at $\xi=0$, say,

$$\sum_{k=0}^{+\infty}\varphi_k''(0,\hbar,\delta)=\lim_{n\to+\infty}\sigma_n\left(=\sum_{k=0}^{n}\varphi_k''(0,\hbar,\delta)\right) \tag{27}$$

must converge, too. We can see

$$\sigma_1=a+(\delta-a)(1+\hbar)-\frac{\hbar}{2\delta}-\frac{3\hbar\alpha}{2\delta}$$

$$\sigma_2=a+(\delta-a)(1+\hbar)^2-\frac{5\hbar^2}{6\delta^3}-\frac{3\hbar^2\alpha}{\delta^3}-\frac{8\hbar^2\alpha^2}{3\delta^3}-\frac{a\hbar^2}{2\delta^2}-\frac{3a\hbar^2\alpha}{2\delta^2}-\frac{\hbar}{\delta}-\frac{3\hbar\alpha}{\delta}$$

DOI: 10.1002/fld

$$\sigma_3 = a+(\delta-a)(1+\hbar)^3-\frac{275\hbar^3}{72\delta^5}-\frac{2303\hbar^3\alpha}{144\delta^5}-\frac{1549\hbar^3\alpha^2}{72\delta^5}-\frac{1417\hbar^3\alpha^3}{144\delta^5}$$
$$-\frac{5a\hbar^3}{2\delta^4}-\frac{9a\hbar^3\alpha}{\delta^4}-\frac{8a\hbar^3\alpha^2}{\delta^4}-\frac{5\hbar^2}{2\delta^3}+\frac{5\hbar^3}{6\delta^3}-\frac{9\hbar^2\alpha}{\delta^3}+\frac{3\hbar^3\alpha}{\delta^3}-\frac{8\hbar^2\alpha^2}{\delta^3}$$
$$-\frac{a^2\hbar^3}{2\delta^3}-\frac{3a^2\hbar^3\alpha}{2\delta^3}-\frac{3a\hbar^2}{2\delta^2}-\frac{9a\hbar^2\alpha}{2\delta^2}+\frac{8\hbar^3\alpha^2}{3\delta^3}-\frac{3\hbar}{2\delta}-\frac{9\hbar\alpha}{2\delta}$$

Note that σ_n contains the term $(\delta-a)(1+\hbar)^n$. Thus, $\hbar$ must belong to a subset of the region $|1+\hbar|\leqslant 1$. Note that in Equations (20) and (21) we have defined $\hbar\neq 0$. Our calculations indicate that the series (27) converges if

$$-2<\hbar<0, \quad \delta\geqslant 2 \tag{28}$$

By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$, which ensures the convergence of the solution series. As pointed out by Liao [6], the appropriate region for $\hbar$ is a horizontal line segment. Our solution series contain the auxiliary parameter $\hbar$. We can choose an appropriate value of $\hbar$ to ensure that the solution series converges. We can investigate the influence of $\hbar$ on the convergence of $F''(0)$, by plotting the curve of it versus $\hbar$, as shown in Figure 1 for some examples by $\delta=3$.

Figure 2 shows the residual error of Equation (5) for $\xi=0$ and $\delta=3$. It can be found that the best value for $\hbar$ in Figure 2(left) is -0.475 (first row of Table I) and in Figure 2(right) is -0.55 (second row of Table I). Graphs of $F(\xi)$, $F'(\xi)$ and $F''(\xi)$ for selected values of the parameters and $\delta=3$ are shown in Figure 3. Another example is shown in Figure 4, for which the best value of $\hbar$ is almost -0.5242 (fifth row of Table I).

The values of $F''(0)$ obtained by HAM with minimum residual error of Equations (5) with best value for $\hbar$ are shown in Table I for $\delta=3$. First column shows the value of $F''(0)$, which was obtained before by Anabtawi and Khuri with four terms of Adomian decomposition method [4].

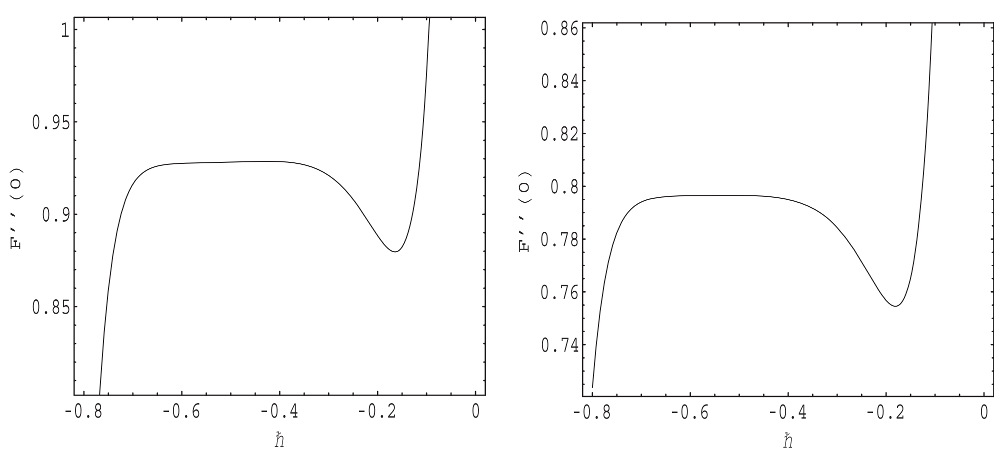

Figure 1. The curves of the $F''(0)$ versus $\hbar$ for the 20th-order approximation. Left: $a=0$, $\alpha=0.5$ and right: $a=0.05$, $\alpha=0.25$.

DOI: 10.1002/fld

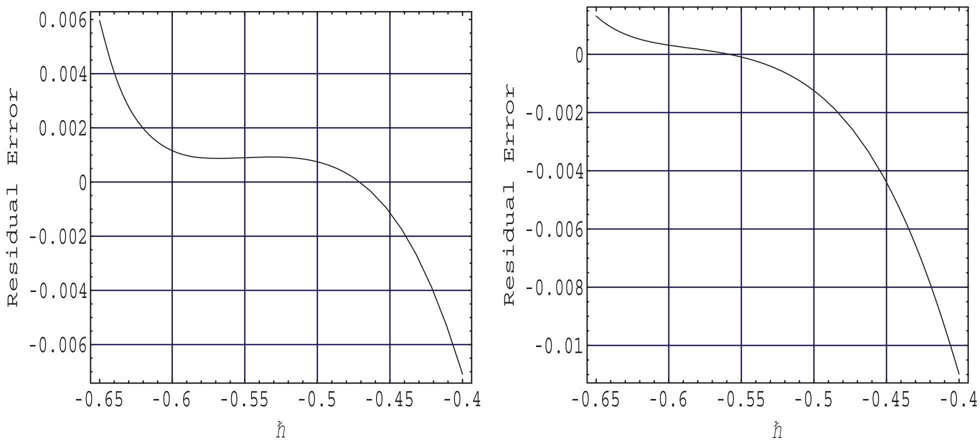

Figure 2. Residual error of (5) versus $\hbar$ for the 20th-order approximation. Left: $a=0$, $\alpha=0.5$ and right: $a=0.05$, $\alpha=0.25$.

Table I. Results for 20th-order of homotopy analysis method for $\delta=3$.

a	α	ADM [4]	Asaithambi method [18]	Shooting method [17]	Crocco's Trans.	HAM	$\hbar$
0	0.5	0.955076	0.92768	0.92772	0.89503	0.928403	−0.475
0.05	0.25	0.849058	0.796295	0.7963	0.74422	0.796528	−0.55
0.02	0.5	0.857887	0.97637	0.976375	0.9395875	0.976905	−0.476
0.01	0.5	0.798723	0.939715	0.93972	0.90602	0.940406	−0.4723
−0.5	0.25	0.354134	0.269049	0.274784	0.306938	0.294572	−0.51
−0.25	0.5	0.491415	0.460428	0.461925	0.478145	0.485848	−0.415
−0.15	0.25	0.380547	0.409882	0.410658	0.420888	0.415261	−0.5242

Third column shows the same values that were obtained by the shooting method described in [17] and the fourth column gives the values calculated by Crocco's transformation. The second column gives these values by the numerical method employed in [18] for the Falkner–Skan equation. In this method, we obtain $F''(0)$ by solving

$$\frac{\mathrm{d}F}{\mathrm{d}\xi}=\eta_\infty U$$

$$\frac{\mathrm{d}U}{\mathrm{d}\xi}=\eta_\infty V$$

$$\frac{\mathrm{d}V}{\mathrm{d}\xi}=-\eta_\infty[F\,V+\alpha(1-U^2)]$$

subject to

$$F(0)=a, \quad U(0)=0, \quad V(0)=\beta$$

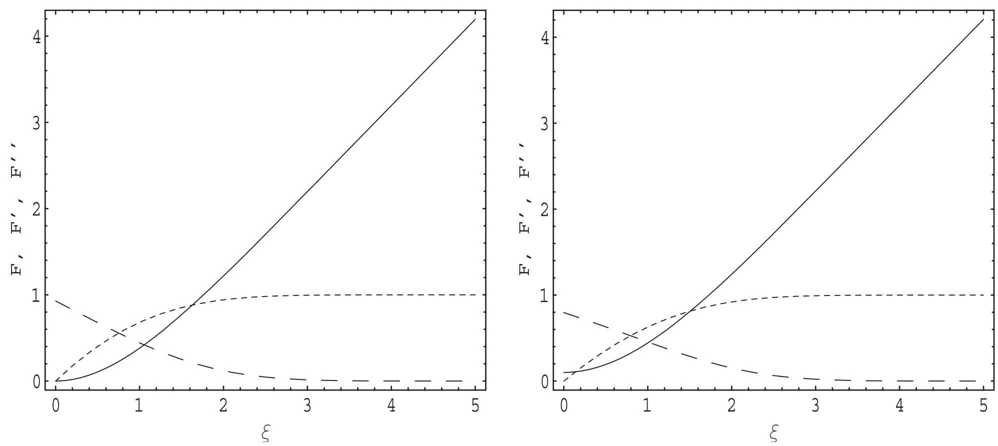

Figure 3. Plot of $F(\xi)$, $F'(\xi)$ and $F''(\xi)$ for the 20th-order approximation. Left: $a=0$, $\alpha=0.5$, $\hbar=-0.475$ and right: $a=0.05$, $\alpha=0.25$, $\hbar=-0.55$.

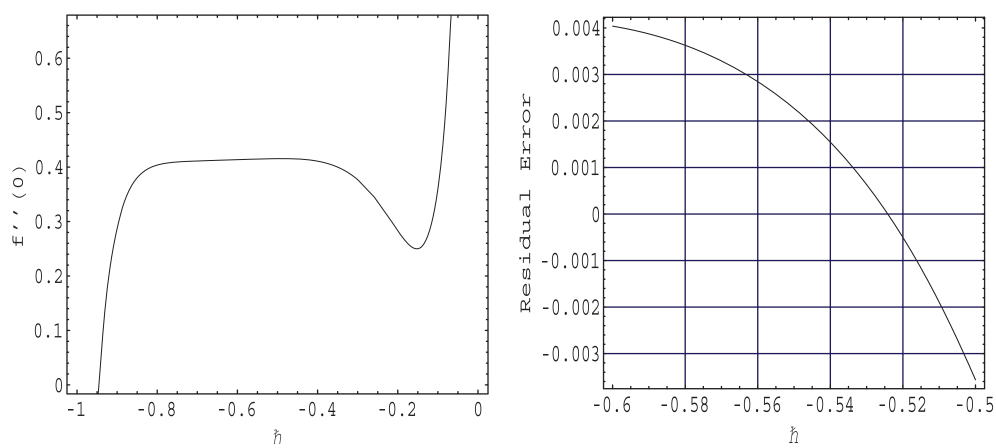

Figure 4. Results for $a=-0.15$, $\alpha=0.25$ and $\delta=3$ in the 20th-order approximation. Left: the $\hbar$-curves of the $F''(0)$ and right: residual error of (5).

with the Runge–Kutta method. Note that β and η_∞ are obtained by using the boundary conditions

$$U(1)=1$$

$$V(1)=0$$

with nested secant iterative method. In this method, the last boundary condition is obtained by using the asymptotic condition $F''(+\infty)=0$.

DOI: 10.1002/fld

6. CONCLUSIONS

In this study the viscoelastic flow of a FENE-P fluid past a plate is analyzed. The order of the resulting non-linear differential system is reduced by one after employing Crocco's transformation. Also for comparison, the third-order non-linear differential problem is solved by a currently developed method, namely the HAM. An interesting flow quantity, $F''(0)$, i.e. the reduced wall heat flux for a porous medium or the skin friction of the wall for a stretched wall is computed numerically and a comparison is shown between the HAM and numerical solutions.

We can obtain the numerical solution of (5) by Adomian decomposition method [4] with HAM while $\hbar = -1$. Our calculations indicate that the best values for $\hbar$ in all cases are not -1 and this shows the affectivity of HAM.

ACKNOWLEDGEMENTS

The first author is thankful to the Higher Education Commission (HEC) of Pakistan for the financial support. The authors would like to thank anonymous referees for valuable suggestions.

REFERENCES

1. Agarwala RP, O'Regan D. Infinite interval problems arising in non-linear mechanics and non-Newtonian fluid flows. *International Journal of Non-linear Mechanics* 2003; **38**:1369–1376.
2. Olagunju DO. A self-similiar solution for forced convection boundary layer flow of a FENE-P fluid. *Applied Mathematics Letters* 2006; **19**:432–436.
3. Asaithambi A. Numerical solution of the Falkner–Skan equation using piecewise linear functions. *Applied Mathematics and Computation* 2004; **159**:267–273.
4. Anabtawi M, Khuri S. On the generalized Falkner–Skan equation governing boundary layer flow of a FENE-P fluid. *Applied Mathematics Letters* 2007; **20**:1211–1215.
5. Alizadeh E, Farhadi M, Sedighi K, Kebria HRE, Ghafourian A. Solution of the Flakner–Skan equation for wedge by Adomian decomposition method. *Communications in Nonlinear Science and Numerical Simulation* 2009; **14**:724–733. DOI: 10.1016/j.cnsns.2007.11.002.
6. Liao SJ. *Beyond Perturbation*: *Introduction to the Homotopy Analysis Method*. Chapman & Hall, CRC Press: London, Boca Raton, 2003.
7. Bataineh AS, Noorani MSM, Hashim I. Solutions of time-dependent Emden–Fowler type equations by homotopy analysis method. *Physics Letters A* 2007; **371**:72–82.
8. Hayat T, Sajid M. On analytic solution for thin film flow of a fourth grade fluid down a vertical cylinder. *Physics Letters A* 2007; **361**:316–322.
9. Hayat T, Sajid M. Analytic solution for axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet. *International Journal of Heat and Mass Transfer* 2007; **50**:75–84.
10. Hayat T, Sajid M. Homotopy analysis of MHD boundary layer flow of an upper-convected Maxwell fluid. *International Journal of Engineering Science* 2007; **45**:393–401.
11. Abbasbandy S, Zakaria FS. Soliton solutions for the fifth-order KdV equation with the homotopy analysis method. *Nonlinear Dynamics* 2008; **51**:83–87.
12. Abbasbandy S. Homotopy analysis method for generalized Benjamin–Bona–Mahony equation. *Zeitschrift für Angewandte Mathematik und Physik* 2008; **59**:51–62.
13. Cheng J, Liao SJ, Mohapatra RN, Vajravelu K. Series solutions of nano boundary layer flows by means of the homotopy analysis method. *Journal of Mathematical Analysis and Applications* 2008; **343**:233–245.
14. Wang J, Chen JK, Liao SJ. An explicit solution of the large deformation of a cantilever beam under point load at the free tip. *Journal of Computational and Applied Mathematics* 2008; **212**:320–330.
15. Xu X, Liao SJ. Dual solutions of boundary layer flow over an upstream moving plate. *Communications in Nonlinear Science and Numerical Simulation* 2008; **13**:350–358.
16. Liao SJ. Notes on the homotopy analysis: some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation* 2009; **14**:983–997. DOI: 10.1016/j.cnsns.2008.04.013.

17. Na TY. *Computational Methods in Engineering Boundary Value Problems*. Academic Press: New York, 1979.
18. Asaithambi NS. A numerical method for the solution of the Falkner–Skan equation. *Applied Mathematics and Computation* 1997; **81**:259–264.
19. Asaithambi A. A second-order finite-difference method for the Falkner–Skan equation. *Applied Mathematics and Computation* 2004; **156**:779–786.
20. Asaithambi A. Solution of the Falkner–Skan equation by recursive evaluation of Taylor coefficients. *Journal of Computational and Applied Mathematics* 2005; **176**:203–214.
21. Chiam TC. Hydromagnetic flow over a surface stretching with a power-law velocity. *International Journal of Engineering Science* 1995; **33**:429–435.

DOI: 10.1002/fld